QUOTA GAMES WITH A CONTINUUM OF PLAYERS

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ABSTRACT

In this paper the notion of m-quota game with a continuum of players is defined and the theory of bargaining sets is generalized to this new class of games. We discuss only the bargaining set M_0 and our results are similar to those obtained in the finite case. Our main result is that for maximal coalition structures the stable payoff functions are exactly those in which almost every non-weak player receives no more than his quota and the weak players receive zero.

In this paper the results that were obtained in [6] for finite m-quota games are generalized to m-quota games with a continuum of players. So this work continues both the study of bargaining sets and of games with a continuum of players. The results that we obtain are somewhat sharper than those in [6], and the proofs are shorter. For an introduction to the subject of games with a continuum of players we refer the reader to [1], which contains also a complete bibliography.

1. **Definitions.** In this section we define the bargaining set M_0 of m-quota games. Our definitions were inspired both by those in [5] of the bargaining set M_0 of finite m-quota games(1), and by the definitions of characteristic function games with a continuum of players in [3] and [7].

We denote by I the unit interval [01]. I is our *set of players*. A *coalition* is a Lebesgue measurable subset of I. A *characteristic function* is a non-negative real function v defined on the set of all coalitions of I. A game is fully described by its characteristic function. Lebesgue measure will be denoted by μ.

DEFINITION 1.1. v is a characteristic function of an *m-quota game*, $0<m<1$, if there exists a measurable function w such that

$$v(S) = \begin{cases} \int_S w d\mu & , \quad \mu(S) = m \\ 0 & , \quad \mu(S) \neq m \end{cases}$$

w is the *quota* of the game; if it exists then it is unique (up to equivalence). The m-quota game with the quota function w will be denoted by (m,w). A *weak* player t is one for whom $w(t) < 0$.

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(1) Which are a modification of the m-quota games of Kalish [4].

Let (m, w) be an m-quota game .A *coalition structure* (c.s.) is a finite set of disjoint coalitions of I, whose union is I. A c.s. is *maximal* (m.c.s.) if it contains a maximal number of m-coalitions (i.e. coalitions whose measure is m). If b is a m.c.s. we denote by $s(b)$ the number of the m-coalitions of b whose intersection with the set of weak players has measure 0.

DEFINITION 1.2. A *coalitionally rational payoff configuration* (c.r.p.c.) is a pair (x, b), where b is a c.s. and x (to be thought of as a payoff vector) is a measurable function that satisfies:

$$\int_B x d\mu = v(B), \quad B \in b,$$

and $\int_S x d\mu \geqq v(S)$, for all coalitions $S, S \subset B \in b$.

If b is a c.s. and S is a coalition we denote by $P(S, b)$ the set of *partners* of S in the c.s. b, i.e. the set $\cup\{B : B \in b, \mu(S \cap B) > 0\}$.

DEFINITION 1.3. Let (x, b) be c.r.p.c. and K and L disjoint, non-null coalitions with the same partners. An *objection* of K against L in (x, b) is a c.r.p.c. (y, c) that satisfies: $\mu(P(K,c) \cap L) = 0$, $y(t) > x(t)$ for almost every $t \in K$ and $y(t) \geqq x(t)$ for almost every $t \in P(K,c)$.

DEFINITION 1.4. Let (x, b) be a c.r.p.c. and (y, c) an objection of a K against an L in (x, b). A *counter objection* of L against K is *a* c.r.p.c. (z,d) that satisfies: $\mu(K - P(L, d)) > 0$, $z(t) \geqq x(t)$ for almost every $t \in P(L, d)$, and $z(t) \geqq y(t)$ for almost every $t \in P(L, d) \cap P(K, c)$.

A c.r.p.c. (x, b) is *stable* if every objection in (x, b) can be countered. The *bargaining set* M_0 is the set of all stable c.r.p.c.'s.

The following two lemmas are not difficult to prove.

LEMMA 1.5. Let $u(t)$ be a real measurable function on a coalition S. If $0 \leqq \theta \leqq \mu(S)$ then there is a coalition $T \subset S$ such that $\mu(T) = \theta$ and $\inf\{u(t): t \in T\} \geqq \sup\{u(t): t \in S - T\}$.

LEMMA 1.6. Let (x, b) be a c.r.p.c., $K = \{t : w(t) > x(t)\}$ *and* $L = \{t : x(t) > \max(0, w(t))\}$. If a coalition $K_1 \subset K$ has an objection (y, c) against a coalition S such that $\int_{K_2}(w - y)d\mu + \int_{K_3}(w - x)d\mu < \int_{S \cap L}(x - w)d\mu$, where $K_2 = \{t : t \in P(K_1, c), w(t) > y(t)\}$ and $K_3 = K - P(K_1, c)$, then S has no counter *objection.*

2. Stable payoff configurations.

THEOREM 2.1. *Let* (m, w) *be an m-quota game,* b *a m.c.s. and* (x, b) *a c.r.p.c. If* $x(t) \leqq \max(0, w(t))$ *a.e. then* $(x, b) \in M_0$.

Proof. We have to show that if (y,c) is an objection of a coalition K against a coalition L, then L has a counter objection. We denote $A=\{t{:}w(t)<0\}$, $L_1=L\cap A$ and $L_2=L-L_1$. $\mu(L_2)=qm+r, 0\leqq r<m$. Let L_3 be a sub-coalition of L_2 whose measure is r. Let L_2–L_3 form q m-coalitions with a quota split; so if $r=0$ L has a counter objection. If $r>0$ let $U\in c$ be an m-coalition; we assert that there is a sub-coalition $U_1\subset U$ such that $\mu(U_1)=m-r$, $\mu(K-U_1)>0$ and $\int_{V_1}(w-y)d\mu\geqq 0$. If $y(t)=w(t)$ for almost every $t\in U$, then we have to delete from U a sub-coalition $U_2\subset U$ whose measure is r and that satisfies $\mu(U_2\cap K)\geqq(r/m)\mu(U\cap K)$ to obtain U_1. If $S=\{t{:}t\in U, y(t)>w(t)\}$ has positive measure, let U_2 be a sub-coalition of S that satisfies $0<\mu(U_2)\leqq r/2$. Denote $V_1=U-U_2$. $\int_{V_1}(w-y)d\mu>0$. If $\mu(K-V_1)=0$ let further U_3 be a sub-coalition of $V_1\cap K$ that satisfies $0<\mu(U_3)\leqq r/2$, and such that $\int_{V_2}(w-y)d\mu>0$, where $V_2=V_1-U_3$. $\mu(K-V_2)>0$. So we can always obtain a coalition $V\subset U$ such that $\int_V(w-y)\,d\mu>0, \mu(V)\geqq m-r$ and $\mu(K-V)>0$. Let U_1 be a sub-coalition of V such that $\mu(U_1)=m-r$ and

$$\inf\{w(t)-y(t):t\in U_1\}\geqq\sup\{w(t)-y(t):t\in V-U_1\}.$$

U_1 has the desired properties. To complete a counter objection of L when $r>0$, L_3 can form an m-coalition F together with U_1. The payments to the members of F will be $w(t)$ for $t\in L_3$ and $y(t)+(1/m-r)\int_{U_1}(w-y)d\mu$, for $t\in U_1$.

A consequence of Theorem 2.1 is

COROLLARY 2.2. *Let (m,w) be an m-quota game and b a m.c.s.; there is always a measurable function x such that the c.r.p.c. $(x,b)\in M_0$.*

THEOREM 2.3. *Let (m,w) be an m-quota game and b a m.c.s. that satisfies $s(b)\geqq 2$. If a c.r.p.c. $(x,b)\in M_0$ then $x(t)\leqq max(0,w(t))$ a.e.,*

Proof. $1=mq+r$, $0\leqq r<m$. *W.l.o.g.* $b=\{B_1,\cdots,B_q,B_{q+1}\}$, $\mu(B_{q+1})=r$ *and* $\mu(A\cap(B_{q-1}\cup B_q))=0$, where $A=\{t{:}w(t)<0\}$. We denote also $A_1=A-B_{q+1}$ *and* $R=B_{q+1}-A$. Let (x,b) be a *c.r.p.c.* We denote $K=\{t:t\in B_j,\ j\leqq q,\ x(t)<w(t)\}$, $J=\{t:t\in B_j,\ j\leqq q,\ w(t)=x(t)\}$ and $L=\{t:x(t)>\max(0,w(t))\}$. We shall prove that the inequality $\mu(L)>0$ implies that $(x,b)\notin M_0$. This will be done by proving the existence of sets U and V such that U has an objection against V, and V has no counter objection. We distinguish the following possibilities:

(a) $$\mu(K)+\mu(J)+\mu(R)<m$$

Let $T\subset L\cup A_1$ be a coalition whose measure is $m-\mu(K)-\mu(R)-\mu(J)$ and that satisfies $\inf\{w(t)-x(t):t\in T\}\geqq\sup\{w(t)-x(t):t\in(L\cup A_1)-T\}$. $P((L\cup A_1)-T,b)=P(K,b)$. We have $\int_{K\cup T}(w-x)d\mu>0$. Since $s(b)\geqq 2$ we have also $\int_{L-T}(x-w)d\mu > \int_T(x-w)d\mu$. K can object against $(L \cup A_1) - T$ by forming, together with $R \cup J \cup T$, an m-coalition F; the payments to the members of F will be $w(t)$ for $t \in R \cup J$, $x(t)$ for $t \in T$ and

$$z(t) = x(t) + \frac{(w(t) - x(t)) \int_{K \cup T}(w-x)d\mu}{\int_K (w-x)\, d\mu}$$

for $t \in K$.
Since $\int_K(w-z)d\mu = \int_K(w-x)d\mu - \int_{K \cup T}(w-x)d\mu = \int_T(x-w)d\mu < \int_{L-T}(x-w)d\mu$ by lemma 1.6 $(L \cup A_1) - T$ has no counter objection.

(b) $$\mu(K) + \mu(R) < m \text{ and } \mu(K) + \mu(R) + \mu(J) \geqq m$$

K can object against $L \cup A_1$ by forming. together with R and a sub-coalition of J whose measure is $m - \mu(R) - \mu(K)$, an m-coalition with a quota split. $L \cup A_1$ has no counter objection.

(c) $$\mu(K) + \mu(R) \geqq m$$

$\mu(K) + \mu(R) = pm + s$, $0 \leqq s < m$. If $s = 0$ K objects against $L \cup A_1$ by forming, together with R, p m-coalitions with a quota split. If $s + \mu(J) \geqq m$ let $Q \subset K$ be a coalition whose measure is s. K can object against $L \cup A_1$ by letting $K - Q$ form together with R p m-coalitions with a quota split, and Q form an additional m-coalition with a quota split together with a sub-coalition of J whose measure is $m - s$. In both cases $L \cup A_1$ cannot counter object. So we may assume in the following that $s > 0$ and $s + \mu(J) < m$. We now show that we may also assume that $\mu(L) \geqq m - s - \mu(J)$. If $\mu(L) < m - s - \mu(J)$ let $Q \subset K$ be a coalition whose measure is s that satisfies

$$\sup\{w(t) - x(t) : t \in Q\} \leqq \inf\{w(t) - x(t) : t \in K - Q\}.\ B_q \subset P(K - Q, b).$$

$K - Q$ can object against $G = P(K - Q, b) - K - J$ by forming, together with R, p m-coalitions with a quota split. $\int_{G \cap L}(x-w)d\mu > \int_{B_q \cap L}(x-w)d\mu = \int_{B_q \cap K}(w-x)d\mu > \int_Q(w-x)d\mu$, so by lemma 1.6 G cannot counter object.
Let now S be a sub-coalition of K whose measure is s that satisfies $\sup\{w(t) - x(t) : t \in S\} \leqq \inf\{w(t) - x(t) : t \in K - S\}$, and T a sub-coalition of $L \cup A_1$ whose measure is $m - s - \mu(J)$ that satisfies $\inf\{w(t) - x(t) : t \in T\} \geqq \sup\{w(t) - x(t) : t \in (L \cup A_1) - T\}$. We distinguish the following sub-cases:

(c.1) $$\int_{S \cup T} (w-x)d\mu \leqq 0$$

In this case there is a coalition $S_1 \subset K$ such that $\mu(S_1) = s, P(K - S_1, b) \supset B_q \cup B_{q-1}$ and $\int_{S_1}(w - x)d\mu < \int_{L \cap (B_q \cup B_{q-1})}(x - w)d\mu$. If $P(K - S, b) \supset B_q \cup B_{q-1}$ we may choose $S_1 = S$. If $P(K - S, b) \not\supset B_{q-1}$ we may assume that $B_{q-1} \cap K = S$. Let $0 < \delta < \min(s, \mu(B_q \cap K))$ and let $U_1 \subset S$ and $U_2 \subset K \cap B_q$ be coalitions whose measure is δ. $S_1 = (S - U_1) \cup U_2$ has the desired properties. Now we can construct an objection of $K - S_1$ against $G = P(K - S_1, b) - K - J$ by letting $K - S_1$ form together with R p m-coalitions with a quota split. By lemma 1.6 G has no counter objection.

(c.2) $$\int_{S \cup T} (w - x)d\mu > 0$$

We have that $P(K, b) = P((L \cup A_1) - T, b)$; also $\int_L (x - w)d\mu > 2 \int_T (x - w)d\mu$ since $\int_T (x - w)d\mu < \int_{B_j \cap L}(x - w)d\mu$ for $j = q - 1, q$. K has the following objection against $(L \cup A_1) - T : K - S$ forms, together with R, p m-coalitions with a quota split, and S joins $J \cup T$ to form an additional m-coalition F; the payments to the members of F will be $x(t)$ for $t \in J \cup T$ and

$$z(t) = x(t) + \frac{(w(t) - x(t)) \int_{S \cup T}(w - x)d\mu}{\int_S (w - x)d\mu}$$

for $t \in S$. Since $\int_{L-T}(x - w)\, d\mu > \int_T (x - w)d\mu = \int_S (w - z)d\mu (L \cup A_1) - T$ cannot counter object.

COROLLARY 2.4. *Let (m, w) be an m-quota game and b a m.c.s. If $w(t) \geqq 0$ a.e. then a c.r.p.c. $(x, b) \in M_0$ if and only if $x(t) \leqq w(t)$ a.e.*

To complete the proof of corollary 2.4 we have to show that if $s(b) = 1$ and $(x, b) \in M_0$ then $x(t) \leqq w(t)$ *a.e.*; we omit the details.

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